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Effect of Interfacial Imperfection on Buckling and Bending Behavior of Composite Laminates

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The problem of modeling elastic behavior of geometrically imperfect laminated composite plates is addressed, with particular attention paid to the effects of interfacial weakness. A spring-layer model is employed to characterize the weakened interfaces, and a von Kármán-type nonlinear theory is developed, taking into account the initial geometric imperfection. The proposed theory includes the recently well-developed third-order zigzag theory as a special case. The displacement-based and mixed governing equations are shown to contain five and four unknowns, respectively. Simple numerical results are given to illustrate the effects of interfacial weakness on the buckling and bending behavior of laminated composite plates.

Introduction

LAMINATED composite structures are widely used in aeronautical and aerospace construction. Although the classical first-order and higher-order overall theories of plates and shells are capable of predicting global responses of elastic structural elements, they are not sufficiently accurate to describe local elastic behavior at the ply level of composite laminated structures. For this reason, various zigzag theories, alternatively termed simplified discrete-layer theories[1] or refined equivalent single-layer theories,[2] recently have been proposed.[3–18] In contrast to overall theories, zigzag theories ensure continuity of both displacements and tractions at layer interfaces, thereby reducing the total number of unknowns, usually to five for displacement-based theories, i.e., the same number as in most overall theories, such as Ref. 19.

A perfect interface, which implies continuity of displacements and tractions across the interface, is assumed in most investigations on composite materials, with the result that interface properties and structures are eliminated. However, in many applications the assumption of a perfect interface is inadequate. Examples for laminated composites could be a coating on the surface of the reinforcing constituent or the presence of a thin layer between adjacent ply layers. A spring-layer model recently has been applied in micromechanics-based research on imperfect interfaces of composites at the reinforcement matrix level.[20–25] However, research on the effects of weak bonding at the ply level of laminated composites appears to be very scarce,[26,27] despite the fact that such materials have been developed for aircraft applications such as the lower wing, fuselage, and tail skins.[28,29]

It has been recognized that small initial geometric imperfections can produce large reductions in the static strength of some structures. Research on flat and curved panels indicates that small initial deviations from perfect geometry can have a significant influence on the buckling behavior.[30–33] However, few attempts have been made to describe such laminated structures with weakened interfaces. To this end, in the present theory for geometrically imperfect laminated plates, the important interfacial properties are incorporated through a spring-layer model as used in micromechanics. An approximate displacement model is proposed that includes displacement jumps across each interface and thus enables interfacial imperfection to be incorporated. As it satisfies the compatibility conditions for transverse shear stresses both at layer interfaces and on the two bounding surfaces of the plate, there is no need for the use of shear correction factors. In the limit of vanishing interface parameters, i.e., for a perfect interface, this theory reduces to the conventional zigzag theory for multilayered anisotropic plates.

Displacement Model

A multilayered plate consisting of k homogeneous anisotropic layers of uniform thickness is shown in Fig. 1. Let $\{x_i\}$ $(i = 1, 2, 3)$ be a Cartesian coordinate system, with the x_3 axis normal to the plane of the plate. A plane parallel to and between the two bounding surfaces of the plate is chosen as the reference plane x_1Ox_2. The bottom surface $(m = 0)$, the $k-1$ interfaces $(m = 1, \ldots, k-1)$, and the top surface $(m = k)$ are denoted by ${}^{(m)}\Omega$ $(m = 0, \ldots, k)$. Thus, the range of the mth layer in the x_3 direction is $[{}^{(m-1)}h, {}^{(m)}h]$, where ${}^{(m)}h$ $(m = 0, \ldots, k)$ is the distance between ${}^{(m)}\Omega$ and the reference plane. Clearly, ${}^{(k)}h - {}^{(0)}h = h$, the total thickness of the plate.

Throughout, a comma followed by a subscript denotes a derivative with respect to the corresponding spatial coordinate. The Einsteinian summation convention applies to repeated subscripts of tensor components, with Latin subscripts ranging from 1 to 3, whereas Greek subscripts are either 1 or 2. The spatial derivative of the Heaviside step function $H(x_3 - {}^{(m)}h)$ with respect to x_3 is stipulated as the right-hand one, so that $H_{,3}(x_3 - {}^{(m)}h) = 0$.

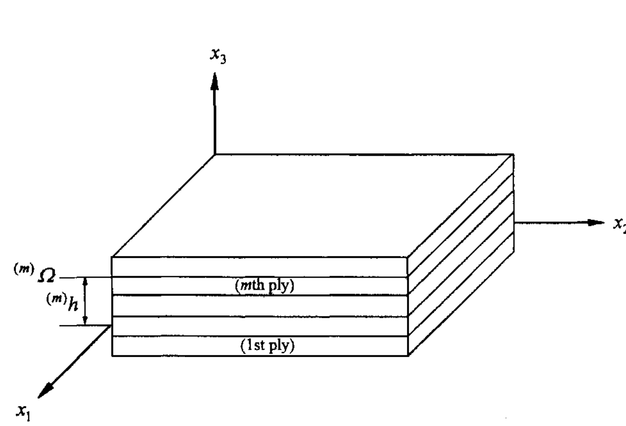

Fig. 1 Geometry of a laminated plate.

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Weak bonding of adjacent layers can be modeled by a mathematical surface across which material properties change discontinuously, with the interfacial tractions being continuous and the displacements being discontinuous. To characterize the imperfect interfaces in the evaluation of composite behavior, one simple approach is to use a linear spring-layer model as follows:

$$\sigma_{\beta 3}\left(x_\alpha, {}^{(m)}h^+\right) = \sigma_{\beta 3}\left(x_\alpha, {}^{(m)}h^-\right) \qquad (m = 1, \ldots, k-1) \quad \text{(1a)}$$

$$\sigma_{33}\left(x_\alpha, {}^{(m)}h^+\right) = \sigma_{33}\left(x_\alpha, {}^{(m)}h^-\right) \qquad (m = 1, \ldots, k-1) \quad \text{(1b)}$$

$${}^{(m)}\Delta v_\alpha = {}^{(m)}R_{\alpha\beta}(x_\rho)\sigma_{\beta 3}\left(x_\alpha, {}^{(m)}h\right) \qquad (m = 1, \ldots, k-1) \quad \text{(2a)}$$

$${}^{(m)}\Delta v_3 = {}^{(m)}R_{33}(x_\rho)\sigma_{33}\left(x_\alpha, {}^{(m)}h\right) \qquad (m = 1, \ldots, k-1) \quad \text{(2b)}$$

$${}^{(m)}\Delta v_j(x_\alpha) \equiv {}^{(m+1)}v_j\left(x_\alpha, {}^{(m)}h\right) - {}^{(m)}v_j\left(x_\alpha, {}^{(m)}h\right) \qquad (m = 1, \ldots, k-1) \quad (3)$$

where σ_{ij} are components of the stress tensor, ${}^{(m)}v_j$ are components of the displacement vector of the mth layer, and ${}^{(m)}R_{\alpha\beta}$ and ${}^{(m)}R_{33}$ in Eq. (2) represent the compliance coefficients of the mth spring-layer interface ${}^{(m)}\Omega$. It is clear from Eq. (2) that a perfect interface corresponds to ${}^{(m)}R_{\alpha\beta} = 0$ and ${}^{(m)}R_{33} = 0$, whereas ${}^{(m)}R_{\alpha\beta} \to \infty$ and ${}^{(m)}R_{33} \to \infty$ represent complete debonding, i.e., $\sigma_{i3} = 0$ on ${}^{(m)}\Omega$. Hence, a slightly weakened interface may be modeled by small values of ${}^{(m)}R_{\alpha\beta}$ and ${}^{(m)}R_{33}$.

When ${}^{(m)}R_{33} = 0$ in conjunction with finite values of ${}^{(m)}R_{\alpha\beta}$, the constitutive relations of the interface allow relative sliding between the two surfaces but no separation. Therefore, the free-sliding case is obtained by setting ${}^{(m)}R_{\alpha\beta} \to \infty$ with ${}^{(m)}R_{33} = 0$. Note that when ${}^{(m)}\Delta v_3 < 0$ this mathematical model results in a physically impossible phenomenon, because one constituent would have to penetrate another.[22,24,25] This violates the compatibility requirements and therefore the model is then apparently unreasonable. However, the normal stress σ_{33} for the plate problem under consideration is assumed to be negligibly small compared with other stress components and so is ignored in the present theory, as in most other plate and shell theories. This automatically leads to an identity equation [Eq. (1b)] and a vanishing displacement jump ${}^{(m)}\Delta v_3$ from Eq. (2b) regardless of the value of the interface parameter ${}^{(m)}R_{33}$. Therefore, it seems reasonable to use this spring-layer model to characterize imperfect bonding in shear.

In the case ${}^{(m)}\Delta v_3 = 0$ $(m = 1, \ldots, k-1)$, the displacement model of the plate can be approximated as

$$v_\alpha(x_i) = u_\alpha + \psi_\alpha x_3 + \varphi_\alpha x_3^2 + \eta_\alpha x_3^3 + \sum_{m=1}^{k-1}\left[{}^{(m)}\Delta v_\alpha + {}^{(m)}u_\alpha\left(x_3 - {}^{(m)}h\right)\right]H\left(x_3 - {}^{(m)}h\right) \quad \text{(4a)}$$

$$v_3(x_i) = u_3 \quad \text{(4b)}$$

where the terms ${}^{(m)}\Delta v_\alpha$, $(m = 1, \ldots, k-1)$, which were excluded for perfect interfaces,[5,6,12,13,15–17] have been retained. These terms imply that displacement discontinuities are allowed at interfaces, thus permitting incorporation of imperfect interfaces for multilayered plates, e.g., weakened bonding. However, a theory developed for calculating delamination[34] needs more terms than Eqs. (4) retain.

The existence of an initial stress-free geometric imperfection, which refers to the transverse displacement u_3^0, is assumed. By convention, the transverse deflection u_3 is measured from the imperfect surface. The small strain, in the sense of moderately small rotations, and stress components of the plate are expressed as[31–33]

$$e_{ij} = \tfrac{1}{2}\left(v_{i,j} + v_{j,i} + v_{3,i}v_{3,j} + v_{3,i}v_{3,j}^0 + v_{3,i}^0 v_{3,j}\right) \quad \text{(5a)}$$

$$\sigma_{\alpha\beta} = H_{\alpha\beta\omega\rho}e_{\omega\rho} \quad \text{(5b)}$$

$$\sigma_{\alpha 3} = 2E_{\alpha 3\omega 3}e_{\omega 3} \quad \text{(5c)}$$

where e_{ij} are components of the strain tensor, $v_3^0 = u_3^0$ is the initial geometric imperfection, E_{ijkl} are components of the elasticity tensor associated with an elastic anisotropic body, and

$$H_{\alpha\beta\omega\rho} = E_{\alpha\beta\omega\rho} - E_{\alpha\beta 33}E_{33\omega\rho}/E_{3333}$$

Equations (5b) and (5c) are valid only under the assumptions that each layer possesses elastic symmetry about its midsurface and that σ_{33} is vanishingly small.[35]

The compatibility conditions of transverse shear stresses on both bounding surfaces of the plate, as well as at the interfaces, are now used to reduce the number of unknowns in Eq. (4a). The absence of tangential tractions on ${}^{(0)}\Omega$ and ${}^{(k)}\Omega$ yields, through Eqs. (4), (5a), and (5c),

$$\psi_\alpha = -u_{3,\alpha} + 3\left(h + {}^{(0)}h\right){}^{(0)}h\eta_\alpha + \frac{{}^{(0)}h}{h}\sum_{m=1}^{k-1}{}^{(m)}u_\alpha$$
$$\varphi_\alpha = -\frac{3}{2}\left(h + 2{}^{(0)}h\right)\eta_\alpha - \frac{1}{2h}\sum_{m=1}^{k-1}{}^{(m)}u_\alpha \quad (6)$$

The condition [Eq. (1a)] for continuously distributed transverse shear stresses at the interfaces by using Eqs. (4), (5a), (5c), and (6) gives

$$\left({}^{(i+1)}E_{\alpha 3\omega 3} - {}^{(i)}E_{\alpha 3\omega 3}\right)\left[3\left(h + {}^{(0)}h - {}^{(i)}h\right)\left({}^{(0)}h - {}^{(i)}h\right)\eta_\omega + \sum_{m=1}^{i}{}^{(m)}u_\omega + \frac{{}^{(0)}h - {}^{(i)}h}{h}\sum_{m=1}^{k-1}{}^{(m)}u_\omega\right] + {}^{(i)}E_{\alpha 3\omega 3}{}^{(i)}u_\omega = 0 \qquad (i = 1, \ldots, k-1) \quad (7)$$

The preceding $2(k-1)$ linear algebraic equations involving the $2(k-1)$ unknowns ${}^{(i)}u_\alpha$ $(i = 1, \ldots, k-1)$ yield the relationship

$${}^{(i)}u_\alpha = {}^{(i)}a_{\alpha\lambda}\eta_\lambda, \qquad (i = 1, \ldots, k-1) \quad (8)$$

in which the ${}^{(i)}a_{\alpha\lambda}$ depend only on the material elasticity properties of each layer and are therefore known constants.

Substitution of Eqs. (6) and (8) into Eq. (4a) yields

$$v_\alpha = u_\alpha - x_3u_{3,\alpha} + f_{\alpha\lambda}\eta_\lambda + \sum_{m=1}^{k-1}{}^{(m)}\Delta v_\alpha H\left(x_3 - {}^{(m)}h\right) \quad (9)$$

$$f_{\alpha\lambda} \equiv f_{\alpha\lambda}(x_3) = \left[3\left(h + {}^{(0)}h\right){}^{(0)}h\delta_{\alpha\lambda} + \frac{{}^{(0)}h}{h}\sum_{m=1}^{k-1}{}^{(m)}a_{\alpha\lambda}\right]x_3 - \left[\frac{3}{2}\left(h + 2{}^{(0)}h\right)\delta_{\alpha\lambda} + \frac{1}{2h}\sum_{m=1}^{k-1}{}^{(m)}a_{\alpha\lambda}\right]x_3^2 + \delta_{\alpha\lambda}x_3^3 + \sum_{m=1}^{k-1}{}^{(m)}a_{\alpha\lambda}\left(x_3 - {}^{(m)}h\right)H\left(x_3 - {}^{(m)}h\right) \quad (10)$$

where $\delta_{\alpha\lambda}$ is the Kronecker delta function.

From Eqs. (2a), (4b), (5a), (5c), and (9), the displacement jump at each interface is

$${}^{(m)}\Delta v_\alpha = {}^{(m)}R_{\alpha\beta}(x_\rho){}^{(m+1)}E_{\beta 3\omega 3}f_{\omega\lambda,3}\left({}^{(m)}h^+\right)\eta_\lambda \quad (11)$$

Substituting this into Eq. (9) gives the approximate displacement expression

$$v_\alpha = u_\alpha - x_3u_{3,\alpha} + h_{\alpha\lambda}\eta_\lambda \quad (12)$$

$$h_{\alpha\lambda} \equiv h_{\alpha\lambda}(x_i) = f_{\alpha\lambda}(x_3) + \sum_{m=1}^{k-1}{}^{(m)}R_{\alpha\beta}(x_\rho){}^{(m+1)}E_{\beta 3\omega 3}f_{\omega\lambda,3}\left({}^{(m)}h^+\right)H\left(x_3 - {}^{(m)}h\right) \quad (13)$$

The fact that the interface parameter ${}^{(m)}R_{\alpha\beta}$ depends on x_ρ implies that the bonding strength at the interface ${}^{(m)}\Omega$ $(m = 1, \ldots, k-1)$ may be nonuniform, i.e., general cases of a small amount of interface weakness are included in the present theory.

Using the displacement expressions of Eqs. (4b) and (12), Eqs. (5) yield the associated strain and stress components, but their explicit forms are not given here.

Displacement-Based Formulation

From the principle of virtual work, the nonlinear static fundamental equations are

$$N_{\alpha\beta,\beta} = 0 \tag{14a}$$

$$M_{\alpha\beta,\alpha\beta} + N_{\alpha\beta}\left(u_{3,\alpha\beta} + u^0_{3,\alpha\beta}\right) + q = 0 \tag{14b}$$

$$P_{\alpha\beta,\beta} - R_\alpha = 0 \tag{14c}$$

and the associated boundary conditions are specified as

$$\begin{aligned} &n_\beta N_{\alpha\beta} \quad \text{or} \quad u_\alpha, \qquad n_\beta\left[M_{\alpha\beta,\alpha} + N_{\alpha\beta}\left(u_{3,\alpha} + u^0_{3,\alpha}\right)\right] \quad \text{or} \quad u_3 \\ &n_\beta P_{\alpha\beta} \quad \text{or} \quad \eta_\alpha, \qquad n_\beta M_{\alpha\beta} \quad \text{or} \quad u_{3,\alpha} \end{aligned} \tag{15}$$

where $q(x_\alpha)$ is an arbitrarily distributed normal load applied to the surface ${}^{(0)}\Omega$ or ${}^{(k)}\Omega$ and

$$[N_{\alpha\beta}, M_{\alpha\beta}, P_{\lambda\beta}, R_\lambda]^T = \int_{{}^{(0)}h}^{{}^{(0)}h+h} [\sigma_{\alpha\beta}, \sigma_{\alpha\beta}x_3, \sigma_{\alpha\beta}h_{\alpha\lambda}, \sigma_{\alpha k}h_{\alpha\lambda,k}]^T \, \mathrm{d}x_3 \tag{16}$$

By using Eqs. (4b), (5), and (12), Eqs. (16) can be rewritten as

$$\begin{bmatrix} N_{\alpha\beta} \\ M_{\alpha\beta} \\ P_{\alpha\beta} \\ R_\alpha \end{bmatrix} = \begin{bmatrix} C^{(1)}_{\alpha\beta\omega\rho} & -C^{(2)}_{\alpha\beta\omega\rho} & C^{(3)}_{\alpha\beta\omega\rho} & C^{(3)}_{\alpha\beta\omega\rho,\rho} \\ C^{(2)}_{\alpha\beta\omega\rho} & -C^{(4)}_{\alpha\beta\omega\rho} & C^{(5)}_{\alpha\beta\omega\rho} & C^{(5)}_{\alpha\beta\omega\rho,\rho} \\ C^{(3)}_{\omega\rho\alpha\beta} & -C^{(5)}_{\omega\rho\alpha\beta} & C^{(6)}_{\alpha\beta\omega\rho} & C^{(7)}_{\alpha\beta\omega} \\ C^{(3)}_{\omega\rho\alpha\beta,\beta} & -C^{(5)}_{\omega\rho\alpha\beta,\beta} & C^{(7)}_{\omega\rho\alpha} & C^{(8)}_{\alpha\omega} \end{bmatrix} \begin{bmatrix} e^0_{\omega\rho} \\ u_{3,\omega\rho} \\ \eta_{\omega,\rho} \\ \eta_\omega \end{bmatrix} \tag{17}$$

where

$$\left[C^{(1)}_{\alpha\beta\omega\rho}, C^{(2)}_{\alpha\beta\omega\rho}, C^{(3)}_{\alpha\beta\nu\rho}, C^{(4)}_{\alpha\beta\omega\rho}, C^{(5)}_{\alpha\beta\nu\rho}, C^{(6)}_{\lambda\beta\nu\rho}, C^{(7)}_{\lambda\beta\nu}\right] = \int_{{}^{(0)}h}^{{}^{(0)}h+h} H_{\alpha\beta\omega\rho}\left[1, x_3, h_{\omega\nu}, x_3^2, x_3h_{\omega\nu}, h_{\alpha\lambda}h_{\omega\nu}, h_{\alpha\lambda}h_{\omega\nu,\rho}\right] \mathrm{d}x_3 \tag{18}$$

$$C^{(8)}_{\lambda\nu} = \int_{{}^{(0)}h}^{{}^{(0)}h+h} \left(H_{\alpha\beta\omega\rho}h_{\alpha\lambda,\beta}h_{\omega\nu,\rho} + E_{\alpha3\omega3}h_{\alpha\lambda,3}h_{\omega\nu,3}\right) \mathrm{d}x_3 \tag{19}$$

$$e^0_{\omega\rho} = \tfrac{1}{2}\left(u_{\omega,\rho} + u_{\rho,\omega} + u_{3,\omega}u_{3,\rho} + u_{3,\omega}u^0_{3,\rho} + u^0_{3,\omega}u_{3,\rho}\right) \tag{20}$$

Finally, substitution of Eqs. (17) into Eqs. (14) yields

$$\begin{aligned} &C^{(1)}_{\alpha\beta\omega\rho}e^0_{\omega\rho,\beta} - C^{(2)}_{\alpha\beta\omega\rho}u_{3,\omega\rho\beta} + \left(C^{(3)}_{\alpha\beta\omega\rho}\eta_\omega\right)_{,\rho\beta} = 0 \\ &C^{(2)}_{\alpha\beta\omega\rho}e^0_{\omega\rho,\alpha\beta} - C^{(4)}_{\alpha\beta\omega\rho}u_{3,\omega\rho\alpha\beta} + \left(C^{(5)}_{\alpha\beta\omega\rho}\eta_\omega\right)_{,\rho\alpha\beta} + \left[C^{(1)}_{\alpha\beta\omega\rho}e^0_{\omega\rho} \right. \\ &\quad \left. - C^{(2)}_{\alpha\beta\omega\rho}u_{3,\omega\rho} + \left(C^{(3)}_{\alpha\beta\omega\rho}\eta_\omega\right)_{,\rho}\right]\left(u_{3,\alpha\beta} + u^0_{3,\alpha\beta}\right) + q = 0 \\ &C^{(3)}_{\omega\rho\alpha\beta}e^0_{\omega\rho,\beta} - C^{(5)}_{\omega\rho\alpha\beta}u_{3,\omega\rho\beta} + C^{(6)}_{\alpha\beta\omega\rho}\eta_{\omega,\rho\beta} \\ &\quad + \left(C^{(6)}_{\alpha\beta\omega\rho,\beta} + C^{(7)}_{\alpha\rho\omega} - C^{(7)}_{\omega\rho\alpha}\right)\eta_{\omega,\rho} + \left(C^{(7)}_{\alpha\beta\omega,\beta} - C^{(8)}_{\alpha\omega}\right)\eta_\omega = 0 \end{aligned} \tag{21}$$

These equations are the displacement-based governing equations, which need to be solved with the boundary conditions of Eqs. (15) to obtain the five unknowns u_α, u_3, and η_α for any set of plate and external load parameters. Equations (21) have variable coefficients because of the nonuniform values of the interface parameters ${}^{(m)}R_{\alpha\beta}$ at the interfaces ${}^{(m)}\Omega$ $(m = 1, \ldots, k-1)$. However, for problems with uniform bonding strength at each interface, Eqs. (21) have constant coefficients. By setting ${}^{(m)}R_{\alpha\beta} = 0$ $(m = 1, \ldots, k-1)$, the corresponding governing equations and boundary conditions become simply those for perfect bonding. In the linear and nonlinear static cases of flat plates with perfect interfaces, the present theory becomes similar to other theories.[5,6,12,13,15–17]

Mixed Formulation

Sometimes it is convenient to introduce a stress function Φ defined by

$$N_{\alpha\beta} = \varepsilon_{\alpha\omega}\varepsilon_{\beta\rho}\Phi_{,\omega\rho} \tag{22}$$

where $\varepsilon_{\alpha\beta}$ is the two-dimensional permutation tensor. With this expression, Eq. (14a) holds automatically, and the first of Eqs. (17) can be rewritten as

$$e^0_{\lambda\nu} = D^{(1)}_{\lambda\nu\omega\rho}\Phi_{,\omega\rho} + D^{(2)}_{\lambda\nu\omega\rho}u_{3,\omega\rho} - \left(D^{(3)}_{\lambda\nu\omega\rho}\eta_\omega\right)_{,\rho} \tag{23}$$

where $D^{(1)}_{\lambda\nu\omega\rho}$, $D^{(2)}_{\lambda\nu\omega\rho}$, and $D^{(3)}_{\lambda\nu\omega\rho}$ satisfy the following relationship:

$$\begin{aligned} C^{(1)}_{\alpha\beta\lambda\nu}D^{(1)}_{\lambda\nu\omega\rho} = \varepsilon_{\alpha\omega}\varepsilon_{\beta\rho}, \qquad C^{(1)}_{\alpha\beta\lambda\nu}D^{(2)}_{\lambda\nu\omega\rho} = C^{(2)}_{\alpha\beta\omega\rho} \\ C^{(1)}_{\alpha\beta\lambda\nu}D^{(3)}_{\lambda\nu\omega\rho} = C^{(3)}_{\alpha\beta\omega\rho} \end{aligned} \tag{24}$$

Substituting Eq. (23) into the remainder of Eqs. (17) gives

$$\begin{aligned} M_{\alpha\beta} &= C^{(2)}_{\alpha\beta\lambda\nu}D^{(1)}_{\lambda\nu\omega\rho}\Phi_{,\omega\rho} + \left(C^{(2)}_{\alpha\beta\lambda\nu}D^{(2)}_{\lambda\nu\omega\rho} - C^{(4)}_{\alpha\beta\omega\rho}\right)u_{3,\omega\rho} \\ &\quad - \left[\left(C^{(2)}_{\alpha\beta\lambda\nu}D^{(3)}_{\lambda\nu\omega\rho} - C^{(5)}_{\alpha\beta\omega\rho}\right)\eta_\omega\right]_{,\rho} \\ P_{\alpha\beta} &= C^{(3)}_{\lambda\nu\alpha\beta}D^{(1)}_{\lambda\nu\omega\rho}\Phi_{,\omega\rho} + \left(C^{(3)}_{\lambda\nu\alpha\beta}D^{(2)}_{\lambda\nu\omega\rho} - C^{(5)}_{\omega\rho\alpha\beta}\right)u_{3,\omega\rho} \\ &\quad - \left(C^{(3)}_{\lambda\nu\alpha\beta}D^{(3)}_{\lambda\nu\omega\rho} - C^{(6)}_{\alpha\beta\omega\rho}\right)\eta_{\omega,\rho} - \left(C^{(3)}_{\lambda\nu\alpha\beta}D^{(3)}_{\lambda\nu\omega\rho,\rho} - C^{(7)}_{\alpha\beta\omega}\right)\eta_\omega \\ R_\alpha &= C^{(3)}_{\lambda\nu\alpha\beta,\beta}D^{(1)}_{\lambda\nu\omega\rho}\Phi_{,\omega\rho} + \left(C^{(3)}_{\lambda\nu\alpha\beta,\beta}D^{(2)}_{\lambda\nu\omega\rho} - C^{(5)}_{\omega\rho\alpha\beta,\beta}\right)u_{3,\omega\rho} \\ &\quad - \left(C^{(3)}_{\lambda\nu\alpha\beta,\beta}D^{(3)}_{\lambda\nu\omega\rho} - C^{(7)}_{\omega\rho\alpha}\right)\eta_{\omega,\rho} - \left(C^{(3)}_{\lambda\nu\alpha\beta,\beta}D^{(3)}_{\lambda\nu\omega\rho,\rho} - C^{(8)}_{\alpha\omega}\right)\eta_\omega \end{aligned} \tag{25}$$

Thus, invoking Eqs. (25) in Eqs. (14b) and (14c) yields

$$\begin{aligned} &C^{(2)}_{\alpha\beta\lambda\nu}D^{(1)}_{\lambda\nu\omega\rho}\Phi_{,\omega\rho\alpha\beta} + \left(C^{(2)}_{\alpha\beta\lambda\nu}D^{(2)}_{\lambda\nu\omega\rho} - C^{(4)}_{\alpha\beta\omega\rho}\right)u_{3,\omega\rho\alpha\beta} \\ &\quad - \left[\left(C^{(2)}_{\alpha\beta\lambda\nu}D^{(3)}_{\lambda\nu\omega\rho} - C^{(5)}_{\alpha\beta\omega\rho}\right)\eta_\omega\right]_{,\rho\alpha\beta} \\ &\quad + \varepsilon_{\alpha\omega}\varepsilon_{\beta\rho}\Phi_{,\omega\rho}\left(u_{3,\alpha\beta} + u^0_{3,\alpha\beta}\right) + q = 0 \\ &C^{(3)}_{\lambda\nu\alpha\beta}D^{(1)}_{\lambda\nu\omega\rho}\Phi_{,\omega\rho\beta} + \left(C^{(3)}_{\lambda\nu\alpha\beta}D^{(2)}_{\lambda\nu\omega\rho} - C^{(5)}_{\omega\rho\alpha\beta}\right)u_{3,\omega\rho\beta} \\ &\quad - \left(C^{(3)}_{\lambda\nu\alpha\beta}D^{(3)}_{\lambda\nu\omega\rho} - C^{(6)}_{\alpha\beta\omega\rho}\right)\eta_{\omega,\rho\beta} + \left(C^{(6)}_{\alpha\beta\omega\rho,\beta} + C^{(7)}_{\alpha\rho\omega} \right. \\ &\quad \left. - C^{(7)}_{\omega\rho\alpha} - C^{(3)}_{\lambda\nu\alpha\beta}D^{(3)}_{\lambda\nu\omega\rho,\beta} - C^{(3)}_{\lambda\nu\alpha\rho}D^{(3)}_{\lambda\nu\omega\beta,\beta}\right)\eta_{\omega,\rho} \\ &\quad + \left(C^{(7)}_{\alpha\beta\omega,\beta} - C^{(8)}_{\alpha\omega} - C^{(3)}_{\lambda\nu\alpha\beta}D^{(3)}_{\lambda\nu\omega\rho,\rho\beta}\right)\eta_\omega = 0 \end{aligned} \tag{26}$$

Elimination of u_α in Eq. (20) yields the compatibility equation that, when combined with Eq. (23), becomes

$$\begin{aligned} &\varepsilon_{\alpha\lambda}\varepsilon_{\beta\nu}\left[D^{(1)}_{\lambda\nu\omega\rho}\Phi_{,\alpha\beta\omega\rho} + D^{(2)}_{\lambda\nu\omega\rho}u_{3,\alpha\beta\omega\rho} - \left(D^{(3)}_{\lambda\nu\omega\rho}\eta_\omega\right)_{,\alpha\beta\rho} \right. \\ &\quad \left. + \tfrac{1}{2}\left(u_{3,\alpha\beta}u_{3,\lambda\nu} + u_{3,\alpha\beta}u^0_{3,\lambda\nu} + u^0_{3,\alpha\beta}u_{3,\lambda\nu}\right)\right] = 0 \end{aligned} \tag{27}$$

Equations (26) and (27) therefore constitute four coupled governing equations in terms of the four unknowns Φ, u_3, and η_α.

Numerical Results

The present theory for geometrically imperfect laminated composite plates is quite general in the sense of von Kármán nonlinearity. Nonlinear static solutions based on the derived governing equations and boundary conditions could be obtained under any transverse and edge loading. However, because of the complexity of the governing equations, only simple numerical examples allowing for closed-form solutions are presented to verify the primary contribution of this theory. Therefore, the following examples are all for a rectangular laminated composite plate, which has identical uniform bonding of interfaces and is simply supported at its edges $x_1 = 0$, a and $x_2 = 0$, b. They are used to examine the effect of weak bonding on overall and local behavior.

For linear buckling problems, conditions ensuring that the prebuckled state of a plate is flat have been given on the basis of the classical theory of laminated composite plates.[36] Stronger conditions were given for a special laminated plate to maintain an initial state of uniform strains e_{11}, e_{22}, and $\sigma_{33} = 0$, using exact three-dimensional elasticity.[37] For simplicity, to enable the present buckling results to be compared with the exact three-dimensional solution,[37] the same example problem has been used, i.e., an orthotropic three-layered plate with ${}^{(1)}h/h = 0.1$ and ${}^{(2)}h/h = 0.9$, and identical relative values of the elastic moduli for each layer:

$$\frac{E_{2222}}{E_{1111}} = 0.543103, \qquad \frac{E_{3333}}{E_{1111}} = 0.530172$$
$$\frac{E_{1122}}{E_{1111}} = 0.23319, \qquad \frac{E_{1133}}{E_{1111}} = 0.010776$$
$$\frac{E_{2233}}{E_{1111}} = 0.098276, \qquad \frac{E_{1212}}{E_{1111}} = 0.262931 \tag{28}$$
$$\frac{E_{1313}}{E_{1111}} = 0.159914, \qquad \frac{E_{2323}}{E_{1111}} = 0.26681$$

For such a square plate subjected to constant uniaxial edge compression N_{11}^0 in the x_1 direction, the buckling mode corresponding to the lowest critical load has the form

$$[u_1, \eta_1] = [U_1, H_1]\cos(\pi x_1/a)\sin(\pi x_2/b)$$
$$[u_2, \eta_2] = [U_2, H_2]\sin(\pi x_1/a)\cos(\pi x_2/b) \tag{29}$$
$$u_3 = U_3 \sin(\pi x_1/a)\sin(\pi x_2/b)$$

The solution for linear bending under the action of transverse pressure $q = q_0 \sin(\pi x_1/a)\sin(\pi x_2/b)$ can be expressed in the same form as Eq. (29). The material chosen for bending computation is a four-ply laminated plate with identical thickness and stiffness properties for each ply:

$$E_L/E_T = 25, \qquad G_{LT}/E_T = 0.5$$
$$G_{TT}/E_T = 0.2, \qquad v_{LT} = v_{TT} = 0.25 \tag{30}$$

where E is the tensile modulus, G is the shear modulus, v is Poisson's ratio, and the subscripts L and T refer to the directions parallel and normal to the fibers, respectively.

With Eqs. (29), exact solutions can easily be obtained for static buckling and bending problems. Numerical results are tabulated in Tables 1 and 2 and are plotted in Fig. 2 by taking ${}^{(0)}h = 0$ and, for each interface, ${}^{(m)}R_{\alpha\beta} = \delta_{\alpha\beta}\bar{R}h/E$, where $\bar{R}$ is a dimensionless quantity, with $E = {}^{(2)}E_{1111}$ for the buckling problem and $E = E_T$ for the bending problem. The transverse shear stresses in Table 2 and Fig. 2 were calculated from the equilibrium equation $\sigma_{\alpha k,k} = 0$, as is normally done,[3,5,12,13,16,17] instead of from the constitutive equations.

Table 1 gives the buckling stress parameter

$$k_x = 12N_{11}^0(b/h)^2/\left[(0.2\beta + 0.8)\left({}^{(2)}E_{1111}h\pi^2\right)\right]$$

for a square plate for different values of the parameters $\bar{R}$ and $\beta = {}^{(1)}E_{1111}/{}^{(2)}E_{1111}$. The exact three-dimensional elasticity solution for perfect bonding[37] also is given for comparison. Table 2 shows the dimensionless central deflection and stresses for bending of a square four-ply plate for various values of $\bar{R}$ and $a/h(= b/h)$, together with comparative exact results[38] for perfect interfaces

Table 1 Buckling stress parameter k_x for a square three-layered plate ($a/h = b/h = 10$)

β	$\bar{R} = 0$ (exact[37])	$\bar{R} = 0$	$\bar{R} = 0.2$	$\bar{R} = 0.4$	$\bar{R} = 0.6$
1	2.770	2.80767	2.79985	2.79193	2.78392
2	3.330	3.38291	3.35824	3.33308	3.30749
5	4.046	4.11833	4.03175	3.94394	3.85536
10	4.200	4.27111	4.10055	3.93208	3.76700
15	4.037	4.09769	3.87430	3.65909	3.45354

Table 2 Central deflection and stresses of a square four-ply (0/90/90/0 deg) symmetric laminated plate under sinusoidal loading[a]

a/h		$\bar{R} = 0$ (exact[38])	$\bar{R} = 0$	$\bar{R} = 0.2$	$\bar{R} = 0.4$	$\bar{R} = 0.6$
4	$\bar{w}$	1.954	1.90601	2.48111	3.08282	3.66623
	$\bar{\sigma}_{11}$	0.720	0.73681	0.88504	1.04440	1.20036
	$\bar{\sigma}_{22}$	0.663	0.70013	0.77606	0.86434	0.96086
	$\bar{\sigma}_{12}$	0.0458	0.04343	0.04940	0.05590	0.06253
	$\bar{\sigma}_{13}$	0.219	0.21093	0.19365	0.17620	0.15819
	$\bar{\sigma}_{23}$	0.292	0.31484	0.28178	0.23683	0.18941
10	$\bar{w}$	0.743	0.73590	0.86150	1.00458	1.16323
	$\bar{\sigma}_{11}$	0.559	0.56107	0.58204	0.60938	0.64260
	$\bar{\sigma}_{22}$	0.401	0.40806	0.44976	0.49188	0.53417
	$\bar{\sigma}_{12}$	0.0276	0.02735	0.02943	0.03163	0.03394
	$\bar{\sigma}_{13}$	0.301	0.30017	0.28870	0.27757	0.26688
	$\bar{\sigma}_{23}$	0.196	0.19954	0.21079	0.21804	0.22125
100	$\bar{w}$	0.4385	0.43460	0.43611	0.43789	0.43995
	$\bar{\sigma}_{11}$	0.539	0.53887	0.53903	0.53923	0.53948
	$\bar{\sigma}_{22}$	0.271	0.27106	0.27178	0.27261	0.27356
	$\bar{\sigma}_{12}$	0.0214	0.02135	0.02139	0.02142	0.02146
	$\bar{\sigma}_{13}$	0.339	0.33879	0.33859	0.33836	0.33810
	$\bar{\sigma}_{23}$	0.139	0.13897	0.13924	0.13955	0.13988

[a]Dimensionless deflection and stresses are defined as follows: $\bar{w} = 100E_T h^3 v_3(a/2, b/2, h/2)/(q_0 a^4)$, $\bar{\sigma}_{11} = h^2\sigma_{11}(a/2, b/2, h)/(q_0 a^2)$, $\bar{\sigma}_{22} = h^2\sigma_{22}(a/2, b/2, (3h/4)^-)/(q_0 a^2)$, $\bar{\sigma}_{12} = h^2\sigma_{12}(0, 0, 0)/(q_0 a^2)$, $\bar{\sigma}_{13} = h\sigma_{13}(0, b/2, h/2)/(q_0 a)$, $\bar{\sigma}_{23} = h\sigma_{23}(a/2, 0, h/2)/(q_0 a)$.

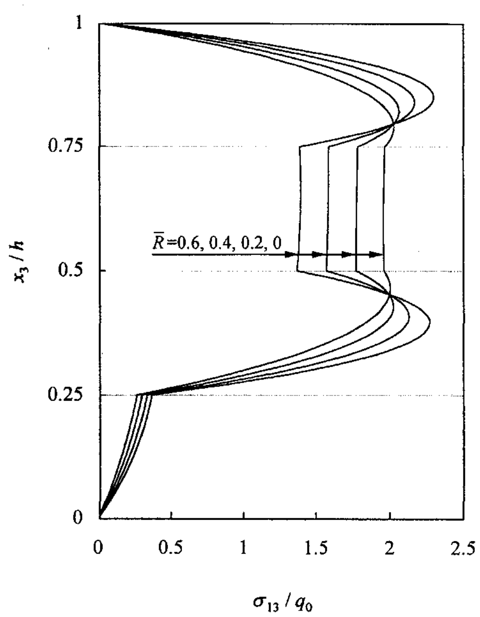

Fig. 2 Through-thickness transverse shear stress at $x_1 = 0$ of an infinitely wide four-ply (90/0/90/0 deg) antisymmetric laminated plate under sinusoidal loading ($a/h = 4$, $b \to \infty$).

calculated from three-dimensional elasticity. Figure 2 shows the through-thickness distribution of dimensionless transverse shear stress. When the theory is used to consider the special case of perfect interfaces, the present results for $\bar{R} = 0$ should be the same as given in Refs. 5 and 13, although some of the present numerical results were not given there. In those papers, comparisons with an exact three-dimensional elasticity solution and several other plate theories confirmed the high accuracy achieved and the necessity of using the third-order zigzag approach. Therefore, assessment of the present theory for the case of perfect bonding is unnecessary.

The buckling stress parameter and nondimensional static central deflection in Tables 1 and 2 show the overall elastic response of plates. As expected, weakening of the interfacial bond causes the rigidity of plates to decrease and hence leads to a decreasing buckling stress parameter or an increasing central deflection for static bending. The variation of stresses in Table 2 and Fig. 2 for bending problems gives a better understanding of the way in which local elastic response is affected by weakened bonding. In practice, the curing process for certain composites is augmented by introducing

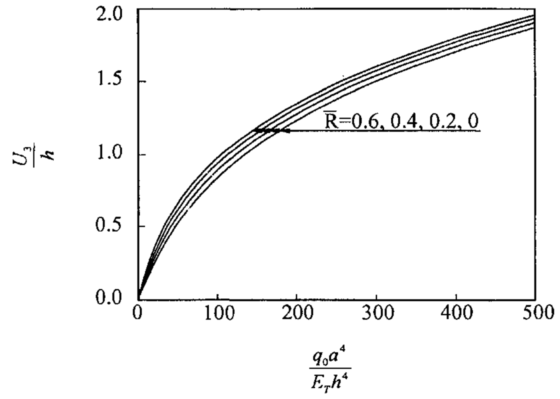

Fig. 3 Nonlinear central deflection–load curves for four-ply (90/0/90/0 deg) antisymmetric laminated square plate under uniform load ($a/h = b/h = 10$).

a very thin adhesive layer in the interfaces to reduce the interlayer stresses.[26] Table 2 and Fig. 2 confirm this important phenomenon, i.e., the maximum interlayer stress for moderately thick and very thick plates decreases significantly as the interfacial parameter increases, as expected.

Because no other corresponding numerical results associated with weak interfacial bonding are available, the accuracy of the proposed analysis could not be explicitly shown in this case. However, the trend of the curves provides a justification that the present results are physically reasonable.

Because an exact solution for nonlinear responses of the laminated plates is impossible, an approximate solution is presented to examine the effect of nonlinearity on a bending problem. By means of the Galerkin technique, a solution for the large deflection of laminated plates under uniform pressure q_0, which satisfies the simply supported boundary conditions, is assumed in the following form:

$$\begin{aligned} u_1 &= U_1 \cos(\pi x_1/a)\sin(\pi x_2/b) \\ &\quad - (\pi/4a)U_3^2 \sin(2\pi x_1/a)\sin^2(\pi x_2/b) \\ u_2 &= U_2 \sin(\pi x_1/a)\cos(\pi x_2/b) \\ &\quad - (\pi/4b)U_3^2 \sin^2(\pi x_1/a)\sin(2\pi x_2/b) \\ u_3 &= U_3 \sin(\pi x_1/a)\sin(\pi x_2/b) \\ \eta_1 &= H_1 \cos(\pi x_1/a)\sin(\pi x_2/b) \\ \eta_2 &= H_2 \sin(\pi x_1/a)\cos(\pi x_2/b) \end{aligned} \tag{31}$$

Use of the Galerkin technique leads to five nonlinear algebraic equations in terms of the five unknowns U_1, U_2, U_3, H_1, and H_2. If U_3 is given first, then the other four unknowns and q_0 can be found by solving five linear algebraic equations. For brevity, the details are omitted, and only final results are presented here. In the special case of perfect bonding, the present results are identical with those of Ref. 8; therefore, it is unnecessary to repeat them in this paper. Figure 3 gives load-deflection curves for different values of the interfacial parameter $\bar{R}$, which indicate that the effect of interfacial imperfection increases with the deflection but not significantly.

Conclusions

A spring-layer model has been used to simulate weak bonding of laminated composite plates with an initial geometric imperfection. By invoking the principle of virtual work and an approximate displacement model, a novel theory for such plates is presented that preserves all the advantages of existing zigzag theories for perfect bonding. The theory features the existence of imperfectly bonded interfaces, with the degree of imperfection spanning from perfect bonding to debonding. The compatibility conditions for transverse shear stresses are satisfied both at layer interfaces and on the two bounding surfaces of the plate. The numbers of unknowns eventually are shown to be five and four in the displacement-based and mixed governing equations, respectively, irrespective of the number of layers. We plan to develop the present work for potential application to delamination problems. As expected, the important problems of global and local behavior of laminated composite plates featuring weak bonding have been solved by the proposed theory.

Because of the complexity of weakened interfaces, which may be caused by fatigue damage or other effects, this paper restricts its attention to a primary understanding of the effect of interfacial damage on responses of composite laminates. Micromechanics-based estimation of interfacial parameters and their effects on the postbuckling behavior of geometrically imperfect laminated plates requires further work.

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